Classification of Plane Congruences in $\mathbf{P}^4_{\mathbf{C}}$ (I)

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Abstract: Nondegenerate plane congruences in the four–dimensional complex projective space with degenerate general focal conic are classified by using the focal method due to Corrado Segre.

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Introduction: The subject of plane congruences in $\mathbf{P}^4_C$ is a classical one. An approach to the classification is due to C. Segre by using the so called (algebro–geometric–differential) theory of foci [8]. An attempt to classification of plane congruences in $\mathbf{P}^4_C$ has been made by M. Italiani in [3] and [4] where he basically uses the more differential geometric tool of E. Cartan's moving frames. Recently, this subject has been revived its application to the study of singularities of theta divisors of Jacobians and to the proof of Torelli's theorem, see [2].

We initially attempted to better understand the classification of plane congruences in $\mathbf{P}^4_C$ given in [4]. We felt that this classification could be clarified by recovering it with the focal method, and this is the aim of this paper. Focal methods were introduced by C. Segre in [7] and [8], and a contemporaneous foundation was given in [1]. The basic definitions and results about plane congruences in $\mathbf{P}^4_C$ and their focal loci can be seen in [2]. A comprehensive exposition of the first order focal locus appeared in [6].

We study the classification of nondegenerate plane congruences in $\mathbf{P}^4_C$ whose focal locus on the general plane consists of a reducible conic. After proving that the degeneration of the conic is equivalent to the existence of developable families of planes contained in the congruence and passing by the plane of the conic (proposition (1.1.2)), we show in 1.2 how the structure of the focal locus determines the focal conic and provides the criterium 1.3 for the classification. Then, the classification is achieved in §2 by doing a very detailed analysis of the first order focal locus. We remark that this is a birrational classification.

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1 Preliminaries.

Throughout this paper, the base field for algebraic varieties is $\mathbf{C}$. Let $\mathbf{P}^4$ be the 4–dimensional complex projective space and $G(2,4)$ the Grassmannian of planes in $\mathbf{P}^4$. We will consider flat families of planes in $\mathbf{P}^4$ parametrized by a quasi–projective irreducible algebraic surface $\Sigma \subset G(2,4)$. They were called plane congruences by classics. We will study the projective generation of the congruence over the open set $U \subset \Sigma$ where the first order focal locus has the expected structure. In this sense our classification is a birrational one and we can suppose that Σ is smooth.

Given $\sigma \in \Sigma$, let $\mathbf{P}^2(\sigma)$ denote the corresponding plane in $\mathbf{P}^4$, and

$$\mathcal{I}_\Sigma := \{(P,\sigma) \in \mathbf{P}^4 \times \Sigma;\ P \in \mathbf{P}^2(\sigma)\}$$

is the Incidence Variety. Consider the two projections $p : \mathcal{I}_\Sigma \longrightarrow \mathbf{P}^4$ and $q : \mathcal{I}_\Sigma \longrightarrow \Sigma$. We call $V(\Sigma) := \overline{p(\mathcal{I}_\Sigma)}$ the projective realization of the plane congruence. The congruence is said to be nondegenerate if p is dominant. From now on all congruences will be nondegenerate and we refer to them with the notation (Σ, p, q).

Recall the definition of focal point and refer to [2] for details: given $\sigma \in \Sigma$ and a point $Q \in \mathbf{P}^2(\sigma)$, the differential map $(dp)_Q : T_{\mathcal{I}_\Sigma,(Q,\sigma)} \longrightarrow T_{P^4,p(Q,\sigma)}$ provides the characteristic map

$$\chi(\sigma) : T_{\Sigma,\sigma} \longrightarrow H^0(\mathbf{P}^2(\sigma), \mathcal{N}_{P^2(\sigma),P^4})$$

and Q is said to be a focal point of the congruence if there is some $w \in T_{\Sigma,\sigma}$ such that $\chi(\sigma)(w)(Q) = 0$. It's well known how all these linear maps globalize to a homomorphism $\lambda : T(p) \longrightarrow \mathcal{N}_{\mathcal{I}_\Sigma, P^4\times\Sigma}$ of sheaves of rank 2, where

$$T(p) := \mathrm{Hom}(\Omega_{\Sigma\times P^4|P^4}, \mathcal{O}_{\Sigma\times P^4})$$

and the first order focal locus of the nondegenerate congruence is well defined as a proper closed subscheme $F_1(\Sigma) \subset \mathcal{I}_\Sigma$ by the condition $\det(\lambda) = 0$.

We can write the equations of the first order focal locus as it was described in [6]. First of all, we can compute $\chi(\sigma)(w)(Q)$ by taking a curve $C \subset \Sigma$ such that $T_{C,\sigma} = \langle w \rangle$ and another curve $D \subset \mathcal{I}_\Sigma$ passing by $(Q,\sigma) \in \mathcal{I}_\Sigma$ such that $q : D \longrightarrow C$ is a local isomorphism at (Q,σ). If $T_{D,(Q,\sigma)} = \langle v \rangle$, then $(dq)_{(Q,\sigma)}(v) = w$ and $\chi(\sigma)(w)(Q) = [(dp)_{(Q,\sigma)}(v)]$, where $[\,]$ denotes the class in $(\mathcal{N}_{P^2(\sigma),P^4})_Q$.

Let $\Delta \subset \mathbf{C}$ be a neighbourhood of the origin and $\sigma : \Delta \longrightarrow C$, $\sigma(0) = \sigma$, a parametrization of C around σ. Consider the induced parametrization $Q : \Delta \longrightarrow D$ around (Q, σ), such that $qQ = \sigma$.If $u \in \Delta$ then $Q(u) \in \mathbf{P}^2(\sigma(u))$, $(Q, \sigma) = (Q(0), \sigma(0))$, $\sigma'(0) = w$ and $Q'(0) = v$. We obtain that Q is a focal point for the direction $w \in T_{\Sigma,\sigma}$ iff $(dp)_{(Q,\sigma)}(v) \in T_{P^2(\sigma),(Q,\sigma)}$, and this shows that the definition of focal point is justly the one given by C. Segre [8].

Given a function $G(u, v)$, let $\frac{\partial G}{\partial u} = G_u$. Suppose $\sigma(u, v)$ a parametrization of Σ around σ, $\sigma(0, 0) = \sigma$. If $w_1 = \frac{\partial \sigma}{\partial u}\big|_{(0,0)}$, $w_2 = \frac{\partial \sigma}{\partial v}\big|_{(0,0)}$, then $T_{\Sigma,\sigma} = \langle w_1, w_2\rangle$. Let $F_1(u, v) = F_2(u, v) = 0$ be the equations of the plane $\mathbf{P}^2(\sigma(u, v))$ in a neighbourhood of $(0, 0)$. Given a direction $\lambda w_1 + \mu w_2$, the plane defined by the equations

$$\lambda F_{1,u}|_{(0,0)} + \mu F_{1,v}|_{(0,0)} = 0, \;\; \lambda F_{2,u}|_{(0,0)} + \mu F_{2,v}|_{(0,0)} = 0$$

is called the plane infinitely near to $\mathbf{P}^2(\sigma)$ for the direction $\lambda w_1 + \mu w_2$, and the focal points correspond to the intersection of the two planes.

We consider the characteristic map $\chi(\sigma) : T_{\Sigma,\sigma} \longrightarrow H^0(\mathbf{P}^2(\sigma), \mathcal{N}_{P^2(\sigma),P^4})$. Since $\mathcal{N}_{P^2(\sigma),P^4)} \cong (\mathcal{O}_{P^2(\sigma)}(1))^2$, if $\chi(\sigma)(w_1) = (f_{11}, f_{12})$ and $\chi(\sigma)(w_2) = (f_{21}, f_{22})$, then the focal points on $\mathbf{P}^2(\sigma)$ corresponding to the direction $(\lambda : \mu)$ are the solutions $Q \in \mathbf{P}^2(\sigma)$ of the equations $\lambda f_{11}(Q) + \mu f_{21}(Q) = 0$, $\lambda f_{12}(Q) + \mu f_{22}(Q) = 0$; i.e.

$$\det \begin{pmatrix} f_{11}(Q) & f_{21}(Q) \\ f_{12}(Q) & f_{22}(Q) \end{pmatrix} = 0$$

Let $U \subset \Sigma$ be the open set consisting of the smooth points $\sigma \in \Sigma$ such that $p(F_1(\Sigma))$ does not contain the plane $\mathbf{P}^2(\sigma)$. If $\sigma \in U$, the first order focal locus on the plane $\mathbf{P}^2(\sigma)$ consists of the conic $f_{11}f_{22} - f_{12}f_{21} = 0$. We denote this conic by $C(\sigma)$, and consider the family $q : F_1(U) \longrightarrow U$. Let $\mathcal{H}^1_{2,0}(\mathbf{P}^4)$ denote the Hilbert Scheme of conics in $\mathbf{P}^4$. Considering the lower semicontinuous function $U \longrightarrow \mathcal{H}^1_{2,0}(\mathbf{P}^4) \longrightarrow \mathbf{N}$ defined by $\sigma \mapsto [C(\sigma)] \mapsto \text{rank}(C(\sigma))$ we obtain the following

Lemma 1.0.1 *Given a nondegenerate plane congruence in* $\mathbf{P}^4$*, either every focal conic is reducible or the general focal conic is irreducible.* ■

We will study the plane congruences whose focal conic is degenerate. These congruences have a geometric characterization that yields a description of their projective generation.

1.1 Developable Families of $\mathbf{P}^r$'s in $\mathbf{P}^n$

Let $\mathbf{P}^n$ be the n–dimensional complex projective space and $G(r, n)$ the Grassmannian parametrizing the r–dimensional subspaces in $\mathbf{P}^n$. A 1–dimensional family of subspaces is said to be a developable family if the general first order

focal locus consists of a $\mathbf{P}^{r-1}$. In order to describe which are the 1–dimensional developable families of planes in $\mathbf{P}^n$, consider a smooth curve $\mathcal{C} \subset G(2,n)$ and its Incidence Variety $\mathcal{I}_\mathcal{C}$ with the two projections $p : \mathcal{I}_\mathcal{C} \longrightarrow \mathbf{P}^n$ and $q : \mathcal{I}_\mathcal{C} \longrightarrow \mathcal{C}$. Let $U \subset \mathcal{C}$ be the open set of elements $\sigma \in \mathcal{C}$ such that the first order focal locus on the plane $\mathbf{P}^2(\sigma)$ is the line $r(\sigma) = \langle Q_1(\sigma), Q_2(\sigma)\rangle$. Write $F_1(\mathcal{C}) := \overline{\bigcup_{\sigma\in U} r(\sigma)}$ the first order focal scheme.

Proposition 1.1.1 *Given a smooth curve $\mathcal{C} \subset G(2,n)$ defining a 1–dimensional developable family of planes, then this family consists of one of the next examples:*

1. *Planes containing a line, if $\dim p(F_1(\mathcal{C})) = 1$.*
2. *If $\dim p(F_1(\mathcal{C})) = 2$, let $F_2(\mathcal{C})$ be the second order focal scheme of $(\mathcal{C}, p, q)$. Then*
   - *(a) If $\dim p(F_2(\mathcal{C})) = 0$, $\mathcal{C}$ consists of the tangent planes to a cone.*
   - *(b) If $\dim p(F_2(\mathcal{C})) = 1$, $\mathcal{C}$ consists of the osculating planes to a curve.*

Proof: Let $U \subset \mathcal{C}$ be the open set of elements $\sigma \in \mathcal{C}$ such that the first order focal locus of $(\mathcal{C}, p, q)$ on the plane $\mathbf{P}^2(\sigma)$ is the line $r(\sigma) = \langle Q_1(\sigma), Q_2(\sigma)\rangle$. We take two curves $D_1, D_2 \subset F_1(\mathcal{C})$ such that $D_i \cap r(\sigma) = Q_i$ and the restriction maps $q : D_i \longrightarrow \mathcal{C}$ are local isomorphisms at Q_i. Let $\lambda Q_1 + \mu Q_2$ be a point in the line $r(\sigma)$. Since this line is a focal line for the family of planes, taking a point $Q_3 \in \mathbf{P}^2(\sigma)$ such that $\mathbf{P}^2(\sigma) = \langle Q_1, Q_2, Q_3\rangle$, we have $\operatorname{rank}(Q_1, Q_2, Q_3, \lambda Q_1' + \mu Q_2') \leq 3$. So we have two possibilities over $r(\sigma)$: either $\operatorname{rank}(Q_1, Q_2, \lambda Q_1' + \mu Q_2') \leq 2$ for every $(\lambda : \mu)$, or $\operatorname{rank}(Q_1, Q_2, \lambda Q_1' + \mu Q_2') \leq 2$ for an only one $(\lambda : \mu)$. In the first case, every point of $F_1(\mathcal{C})$ is a focal point, thus the differential map $(dp)_Q : T_{F_1(\mathcal{C}),(Q,\sigma)} \longrightarrow T_{P^4,p(Q,\sigma)}$ is not injective and $\dim p(F_1(\mathcal{C})) < 2$. So the projective realization of the focal family is a line, yielding that our family of planes must be a 1–dimensional family of planes containing a line. We remark that this happens just when every first order focal point is a second order focal point. For the second case, suppose $P_1 \in r(\sigma)$ is the second order focal point in the general line. Consider the set $F_2(\mathcal{C}) \subset F_1(\mathcal{C})$ of such points. This is a proper closed subset and we have two possibilities:

- *$p(F_2(\mathcal{C}))$ is a point.* In this case, the projective realization of $F_1(\mathcal{C})$ is a cone of vertex P_1, and its generators can be parametrized in the way $\langle P_1, P_2(u)\rangle$. So the family of planes is of the form $\langle P_1, P_2(u), P_3(u)\rangle$. Now, the point P_2 is focal for this family, so $\operatorname{rank}(P_1, P_2, P_3, P_2') \leq 3$, but it is not focal for the family of lines, so $\operatorname{rank}(P_1, P_2, P_2') = 3$ and $\mathbf{P}^2(\sigma) = \langle P_1, P_2, P_2'\rangle$. That is, *our family is the family of tangent planes to a cone (consisting of its foci of first order, and with vertex its focus of the second order).*

- $p(F_2(\mathcal{C}))$ *is a curve.* Since rank $(P_1, P_2, P_1') = 2$, the lines in $F_1(\mathcal{C})$ can be written as $\mathbf{P}^1(\sigma(u)) = \langle P_1(u), P_1'(u)\rangle$. Analogously, one can write the family of planes $\mathbf{P}^2(\sigma(u)) = \langle P_1, P_1', P_1''\rangle$. So $\mathcal{C}$ *is parametrizing the osculating planes to the curve consisting of its foci of the second order, and $\mathcal{C}$ is the family of tangent planes to the surface of its foci of the first order.* ■

The first step for giving a Criterium to classify the plane congruences in $\mathbf{P}^4$ with degenerate focal conic is the next result:

Proposition 1.1.2 *Let (Σ, p, q) be a nondegenerate congruence in $\mathbf{P}^4$. There is a developable family passing by the general plane $\mathbf{P}^2(\sigma)$ if and only if the general focal conic $C(\sigma) \subset \mathbf{P}^2(\sigma)$ is degenerate.*

Proof: Since the "only if" part is clear, we only prove the "if" part. Suppose that $C(\sigma)$ is a degenerate conic. If $(\lambda : \mu) \in \mathbf{P}^1$, let $C(\sigma, \lambda : \mu) \subset C(\sigma)$ be the set of focal points corresponding to the direction $(\lambda : \mu)$. If $C(\sigma, \lambda : \mu)$ is a point for all $(\lambda : \mu)$, we consider the incidence variety $\mathcal{I} := \{(P, (\lambda : \mu)) \in \mathbf{P}^2(\sigma) \times \mathbf{P}^1; P \in C(\sigma, \lambda : \mu)\}$ with the two projections $\bar{p} : \mathcal{I} \longrightarrow \mathbf{P}^2(\sigma)$ and $\bar{q} : \mathcal{I} \longrightarrow \mathbf{P}^1$, and we see that $\bar{p}(\mathcal{I}) = C(\sigma)$ would be irreducible. Besides, the function $\mathbf{P}(T_{\Sigma,\sigma}) \cong \mathbf{P}^1 \longrightarrow \mathbf{N}$ defined by $(\lambda : \mu) \mapsto \dim \bar{q}^{-1}(\lambda : \mu)$ is upper semicontinuous and

$$D_\sigma := \{(\lambda : \mu); \dim \bar{q}^{-1}(\lambda : \mu) \geq 1\}$$

is a closed subset in $\mathbf{P}^1$ defined by two quadratic polynomial equations. So, either D_σ is a finite subset of degree 1, 2; or $D_\sigma \cong \mathbf{P}^1$. These solutions correspond to developable families passing by the plane $\mathbf{P}^2(\sigma)$: they are the curves such that the tangent line at the general point corresponds to $(\lambda : \mu) \in \mathbf{P}(T_{\Sigma,\sigma}) \cong \mathbf{P}^1$. ■

Remark: The proof of the proposition shows that the developable families passing by $\mathbf{P}^2(\sigma)$ correspond to the directions in $T_{\Sigma,\sigma}$ whose focal locus is a line. Moreover it's clear that there are four possibilities: there is just one developable family passing by the general plane; there are two different or coincident developable families passing by the general plane; and finally, every 1–dimensional family passing by the general plane is developable.

1.2 Projective Generation of the Focal Conic.

We will show how the focal conic arises in relation to the directions $(\lambda : \mu)$. We observe that the focal conic is generated by the two pencils of lines in $\mathbf{P}^2(\sigma)$ corresponding to the traces of the two pencils of hyperplanes $\lambda f_{11} + \mu f_{21} = 0$ and $\lambda f_{12} + \mu f_{22} = 0$. We have the following possibilities:

- **Only one of the pencils degenerates on a line not belonging to the other pencil**: In this case, for example, there is some $a \neq 0$ such

that $f_{11} = af_{21}$; f_{12} and f_{22} are not proportional and f_{21} is not a linear combination of f_{12} and f_{22}. The focal conic is $f_{21}(af_{12} - f_{22}) = 0$ where $af_{12} - f_{22} = 0$ is the focal line corresponding to the direction $(a : -1)$ and for other direction $(\lambda : \mu) \neq (a : -1)$, the focal locus consists of the point $\lambda f_{12} + \mu f_{22} = f_{21} = 0$. So, we only have one developable system and the focal loci are disjoint.

- **The two pencils degenerate on two lines**: In this case there is some $a \neq 0$ and some $b \neq 0$ such that $f_{11} = af_{21}$, $f_{12} = bf_{22}$, and the focal conic is $f_{21}f_{22} = 0$. We can suppose $a \neq b$ since, in other case, every point of the plane would be focal for $(\lambda : \mu) = (a : -1)$. First, we suppose f_{21} and f_{22} are not proportional. We get the focal line $f_{22} = 0$ corresponding to $(\lambda : \mu) = (a : -1)$ and the focal line $f_{21} = 0$ corresponding to $(\lambda : \mu) = (b : -1)$. For other values of $(\lambda : \mu)$, the focal locus is the intersection point $f_{21} = f_{22} = 0$. So we have two different developable systems and a focal point for every direction. Finally, if f_{21} and f_{22} are proportional, the focal conic consists of the double line $(f_{21})^2 = 0$, and every point is focal for every direction.

- **Two pencils with different base points**: We call Q_1 and Q_2 the base points of the two pencils H_{Q_1} and H_{Q_2}. Since the conic is degenerate, there is a value $(\lambda : \mu)$ such that the equations $\lambda f_{11} + \mu f_{21} = 0$ and $\lambda f_{12} + \mu f_{22} = 0$ give the same line, $\langle Q_1, Q_2 \rangle$. Suppose that this value is $(1 : 0)$, there is some $a \neq 0$ such that $f_{12} = af_{11}$ and the focal conic has equation $f_{11}(f_{22} - af_{21}) = 0$. We get the focal line $f_{11} = 0$ corresponding to $(\lambda : \mu) = (1 : 0)$, and if $(\lambda : \mu) \neq (1 : 0)$, the focal locus is the solution of $\lambda f_{11} + \mu f_{21} = f_{22} - af_{21} = 0$. The focal points are on the line $f_{22} - af_{21} = 0$ but not on $f_{11} = 0$. This case is analogous to the first one: there is only one developable system passing by the general plane and the focal loci are disjoint.

- **Two pencils with the same base point, and one of the pencils can be degenerate**. Suppose that the first pencil H_1 is nondegenerate. If $f_{12} = \lambda_1 f_{11} + \mu_1 f_{21}$, $f_{22} = \lambda_2 f_{11} + \mu_2 f_{21}$, the focal conic is defined by the equation $\lambda_2 f_{11}^2 + (\mu_2 - \lambda_1) f_{11} f_{21} - \mu_1 f_{21}^2 = 0$. The base point is focal for every direction $(\lambda : \mu)$ except for $(\lambda : \mu) = (\lambda\lambda_1 + \mu\lambda_2 : \lambda\mu_1 + \mu\mu_2)$. That's to say, except for the eigenvectors of the matrix $A = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{pmatrix}$. We have the following possibilities:

  - If H_2 is degenerate, then A is degenerate and, choosing a value $(\lambda_0 : \mu_0) \in \mathbf{P}^1$ such that $\lambda_0\lambda_1 + \mu_0\lambda_2 = \lambda_0\mu_1 + \mu_0\mu_2 = 0$, we get the line $\lambda_0 f_{11} + \mu_0 f_{21} = 0$. Taking other different value $(\lambda' : \mu')$, we get the line $\lambda' f_{11} + \mu' f_{21} = 0$ and the focal conic is $(\lambda_0 f_{11} + \mu_0 f_{21})(\lambda' f_{11} + \mu' f_{21}) = 0$.

- If H_2 is nondegenerate, we have two possibilities: either both pencils have common two different values $(\lambda : \mu)$, the eigenvectors of A, and the focal conic is as above; or they have in common all the values, and thus $\mu_1 = \lambda_2 = 0$ and $\lambda_1 = \mu_2$ and every point of the plane is a focal point.
- Finally, it is possible that, being H_2 nondegenerate, the matrix A has a unique eigenvector $(\lambda_0 : \mu_0)$. Then the focal conic is the double line $(\lambda_0 f_{11} + \mu_0 f_{21})^2 = 0$. In this case $(\lambda_0 : \mu_0)$ is a double developable direction and, for the other values, the focal locus consists of the point $f_{11} = f_{21} = 0$. Now, the existence of this focal conic is equivalent to the condition $(\mu_2 - \lambda_1)^2 + 4\mu_1\lambda_2 = 0$.

1.3 Criterium for the Classification.

If (Σ, p, q) is a nondegenerate plane congruence such that the general focal locus is a degenerate conic, the congruence belongs to one of the following types:
α–**Congruence:** There is only one developable system passing through the general plane. Equivalently, the general focal conic consists of two different lines $r \vee r'$ and the focal loci corresponding to all directions are disjoint. For only one direction we get the focal line r, and for other different one we get a point in $r' \setminus (r \cap r')$.
β–**Congruence:** There are two different developable systems passing through the general plane. Equivalently, the general focal conic consists of two different lines $r \vee r'$ with r and r' corresponding to two different developable directions. For other direction different from these ones, the focal locus consists of the singular point $r \cap r'$ which is a focal point for every direction.
γ–**Congruence:** There are two coincident developable systems passing through the general plane. Equivalently, the general focal conic consists of a double line r^2, with r the focal locus for a double developable direction and for all other directions, the focal locus is a fixed point P, which is a focal point for all of them.
δ–**Congruence:** Every 1–dimensional family passing through the general plane of the congruence is developable. This is equivalent to the property that the general focal conic is a double line r^2, being r a focal line for every direction.

2 The Classification.

2.1 δ–Congruences

Let (Σ, p, q) be a δ–congruence and consider the open set $U \subset \Sigma$ of $\sigma \in U$ satisfying $C(\sigma) = r(\sigma)^2$, where $r(\sigma)$ focal line for every direction.

Theorem 2.1.1 *If (Σ, p, q) is a δ–congruence, it consists of the linear system*

of planes containing a line $r \subset \mathbf{P}^4$. This line is the focal locus of the congruence on every plane.

Proof: We define a morphism $\psi : U \longrightarrow G(1,4)$ by $\psi(\sigma) = [r(\sigma)]$ and we will prove that $\Sigma' := \overline{\psi(U)}$ is of dimension 0. Suppose $\dim \Sigma' = 2$. If $r(\sigma) \subset \mathbf{P}^2(\sigma)$ is a focal line and $x \in r(\sigma)$ a general point, we take parametrizations $x, y, z : \Delta \subset \mathbf{C}^2 \longrightarrow \mathcal{I}_U$ and $\sigma : \Delta \longrightarrow U$ such that $\sigma(0,0) = \sigma$, $x(0,0) = x$, $r(\sigma(u,v)) = \langle x(u,v), y(u,v) \rangle$, and $\mathbf{P}^2(\sigma(u,v)) = \langle x(u,v), y(u,v), z(u,v) \rangle$. Since x is a focal point for every direction, $\operatorname{rank}(x, y, z, \lambda x_u + \mu x_v) = 3$ for every $(\lambda : \mu)$. So there is $(\lambda_0 : \mu_0)$ such that x is a focal point for the family of lines $\{r(\sigma)\}_{\sigma \in U}$ for the corresponding direction. But, in particular, this implies that $\dim p(F_1(U)) < \dim F_1(U)$ and $p(F_1(U))$ must be a plane. This is false since every developable family parametrized by a curve in U would consist of the tangent planes to a 1–dimensional family of lines in the plane. Now, suppose that $\dim \Sigma' = 1$. In this case $p(F_1(U))$ is a ruled surface and there is an infinity of planes in the congruence passing through every generator. We can take a curve $\gamma \subset U$ such that $\overline{\psi(\gamma)} = \Sigma'$. Since the focal loci of the planes parametrized by γ consists of the generators of the ruled surface $p(F_1(U))$, by proposition (1.1.1), γ will be the family of tangent planes to $p(F_1(U))$, and so $\dim U = 1$. This is false too. ■

2.2 β/γ–Congruences

Let (Σ, p, q) be a β/γ–congruence. Recall that, if $\sigma \in \Sigma$ is a general point, there is only one point $P(\sigma) \in C(\sigma)$ that is a focal point for every direction. Equivalently, $\operatorname{rank}(dp)_{P(\sigma)} = 2$. Consider the closed subset $S(U) := \{(P, \sigma) \in \mathcal{I}_U : \operatorname{rank}(dp)_{P(\sigma)} \leq 2\} \subset \mathcal{I}_U$. We can suppose, by restricting U if necessary, that $\rho^{-1} = q|_{\overline{S(U)}} \longrightarrow U$ is an isomorphism. Consider $\phi := p\rho$, $\phi(\sigma) = P(\sigma)$, and let $\Sigma' := \overline{\phi(U)}$. For a general point $\sigma \in U$, $(d\phi)_\sigma$ is surjective, yielding $T_{\Sigma', P(\sigma)} \subset T_{P^2(\sigma), P(\sigma)}$. There are three possibilities in according to the dimension of Σ'.

β_1/γ_1–**Congruences:** $\dim \Sigma' = 2$. In this case, the congruence (Σ, p, q) consists of the tangent planes to the surface Σ'. We study how the intrinsic geometry of Σ' provides a β or γ–congruence.

Definition 2.2.1 *A curve C on a surface S is said to be an Asymptotic Curve, when the osculating plane to C at P, $T_{2,C,P}$, and the tangent plane to S at P, $T_{S,P}$, are coincident for every point of C. Two 1-dimensional families of curves γ and γ' on a surface S are said to be a Conjugate Double System if the ruled surface of the tangent lines to the curves of the family γ' at the points of a curve $C \in \gamma$ is developable; that is, the general focal locus is a point.*

Remark 2.2.1 *If $x(u,v)$ denotes a parametrization of S around a general point, it's well known that (see [5]):*

1. *there is a* 1*–dimensional family of asymptotic curves on* S *with differential equation* $dv - \lambda du = 0$ *if and only if there is a function* $\lambda(u,v)$ *such that* $\mathrm{rank}(x, x_u, x_v, x_{uu} + (2\lambda)x_{uv} + \lambda^2 x_{vv}) = 3$ *for every* (u,v)*,*

2. *there is a conjugate double system on* S *with differential equation* $(dv - \lambda du)(dv - \mu du) = 0$ *if and only if there are functions* $\lambda(u,v)$ *and* $\mu(u,v)$ *such that* $\mathrm{rank}(x, x_u, x_v, x_{uu} + (\lambda+\mu)x_{uv} + \lambda\mu x_{vv}) = 3$ *for every* (u,v)*.*

Theorem 2.2.1 *A* β_1/γ_1*–congruence* (Σ, p, q) *consists of the tangent planes to a nondegenerate irreducible surface* $\Sigma' \subset \mathbf{P}^4$*, and either* Σ' *has a conjugate double system and, equivalently,* (Σ, p, q) *is a* β_1*–congruence, or* Σ' *only has a* 1*–dimensional family of asymptotic curves and* (Σ, p, q) *is a* γ_1*–congruence.*

Proof: We lift the parametrization $x(u,v)$ of Σ' to a parametrization $\sigma : \Delta \subset \mathbf{C}^2 \longrightarrow U$ such that $\mathbf{P}^2(\sigma(u,v)) = \langle x(u,v), x_u(u,v), x_v(u,v)\rangle$ is the tangent plane to Σ' at the point $P(\sigma) = x(u,v)$. The condition in Remark (2.2.1.1) is equivalent to $(\frac{\partial}{\partial u} + \lambda\frac{\partial}{\partial v})(x_u + \lambda x_v) \in \mathbf{P}^2(\sigma(u,v))$, and Σ' has a 1–dimensional family of asymptotic curves iff $x_u + \lambda x_v$ is a focal point for $(\frac{\partial}{\partial u} + \lambda\frac{\partial}{\partial v})$ on every plane. Analogously, Σ' has a conjugate double system iff $x_u + \lambda x_v$ is a focal point for $(\frac{\partial}{\partial u} + \mu\frac{\partial}{\partial v})$ and, symmetrically, $x_u + \mu x_v$ is a focal point for $(\frac{\partial}{\partial u} + \lambda\frac{\partial}{\partial v})$. Since $P(\sigma) = x$ is a focal point for every direction, the focal conic consists of $\langle x, x_u + \lambda x_v\rangle \vee \langle x, x_u + \mu x_v\rangle$, and (Σ, p, q) will be a β_1–congruence.
If Σ' has a 1–dimensional family of asymptotic curves, $\langle x, x_u + \lambda x_v\rangle$ is a focal line on every plane. If (Σ, p, q) is a β_1–congruence, there is a point $x_u + \lambda' x_v$, $\lambda' \neq \lambda$ which is a focal point for $(\frac{\partial}{\partial u} + \mu'\frac{\partial}{\partial v})$. By the condition in Remark (2.2.1.2), there would be a third focal point $x_u + \mu' x_v$, then $\mu' = \lambda'$. So the differential equation $dv - \lambda' du = 0$ provides another 1–dimensional family of asymptotic curves on Σ'. We prove that if $\Sigma' \subset \mathbf{P}^4$ is a surface with two families of asymptotic curves, the congruence Σ of tangent planes to Σ' is degenerate. In fact, for the general point $\sigma \in \Sigma$, every point in $\mathbf{P}^2(\sigma)$ is a focal point. Suppose a parametrization of Σ' such that the asymptotic curves have equations $dv = 0$, $du = 0$, then $x_{uu}, x_{vv} \in \langle x, x_u, x_v\rangle$ for every (u,v) and every point $Bx_u + Cx_v \in \langle x_u, x_v\rangle$ is a focal point for the direction $(-B\frac{\partial}{\partial u} + C\frac{\partial}{\partial v})$. ■

β_2/γ_2–**Congruences:** $\dim \Sigma' = 1$**.** In this case, $\dim \phi^{-1}(P(\sigma)) = 1$ for the general $P(\sigma) \in \Sigma'$ and the congruence (Σ, p, q) consists of 1–dimensional families of planes passing through the tangent lines to Σ'.

Theorem 2.2.2 *The congruence* (Σ, p, q) *is a* β_2/γ_2*–congruence iff it consists of* 1*–dimensional families of planes passing through the tangent lines to a curve* Σ'*.* (Σ, p, q) *is a* γ_2*–congruence iff all the families containing the tangent line are linear ones and they contain the corresponding osculating plane to* Σ'*.*

Proof: We observe that there is a developable family $\gamma_{P(\sigma)} = \{\tau \in U : \phi(\tau) = P(\sigma)\}$ passing through the plane $\mathbf{P}^2(\sigma)$. We study the existence of other

different or coincident developable system passing through $\mathbf{P}^2(\sigma)$. The idea consists of taking a plane π that does not meet the curve Σ' and analyzing the intersection with the general plane of (Σ, p, q) in order to be able to parametrize the congruence by the points of π. Considering a general plane $\pi \subset \mathbf{P}^4$, there is an open subset $U \subset \Sigma$ such that $\mathbf{P}^2(\sigma) \cap \pi$ is a point for $\sigma \in U$. We have a dominant morphism $\psi : U \longrightarrow \pi$, $\psi(\sigma) = \mathbf{P}^2(\sigma) \cap \pi$ and $\mathbf{P}^2(\sigma) = \langle T_{\Sigma',\phi(\sigma)}, \psi(\sigma)\rangle$. Since ψ is a local isomorphism at the general point $\sigma \in U$, we can take a parametrization $\sigma(u,v)$ of U around σ and a reference system on π such that $\sigma(0,0) = \sigma$; $\psi(\sigma(u,v)) = (u : v : 1)$. Let $C_{P(\sigma)} = \psi(\gamma_{P(\sigma)})$ the curves satisfying the differential equation $dv - kdu = 0$, for some function $k(u,v)$, around the point $\psi(\sigma)$. The tangent vectors at every point are $\frac{\partial}{\partial u} + k\frac{\partial}{\partial v}$ and $(d\phi)_\sigma(\frac{\partial \sigma}{\partial u} + k\frac{\partial \sigma}{\partial v}) = 0$, then we complete it with $\frac{\partial}{\partial v}$ to get a base of $T_{\pi,\psi(\sigma)}$, and $\mathbf{P}^2(\sigma(u,v)) = \langle \phi(\sigma(u,v)), \frac{\partial(\phi\sigma)}{\partial v}, \psi(\sigma(u,v))\rangle$. There is a developable direction different from $\frac{\partial}{\partial u} + k\frac{\partial}{\partial v}$ in $T_{\Sigma,\sigma}$ iff there are λ and $(a : b) \in \mathbf{P}^1$ such that: $\mathrm{rank}(\phi(\sigma), \frac{\partial(\phi\sigma)}{\partial v}, \psi(\sigma), (\lambda(\frac{\partial}{\partial u} + k\frac{\partial}{\partial v}) + \frac{\partial}{\partial v})(a\frac{\partial(\phi\sigma)}{\partial v} + b\psi(\sigma))) \leq 3$. Equivalently, there is λ such that

$$\mathrm{rank}(\psi(\sigma), \phi(\sigma), \frac{\partial(\phi\sigma)}{\partial v}, \frac{\partial^2(\phi\sigma)}{\partial v^2}, \lambda(\frac{\partial(\psi\sigma)}{\partial u} + k\frac{\partial(\psi\sigma)}{\partial v}) + \frac{\partial(\psi\sigma)}{\partial v}) \leq 4$$

That is: there is a line $\langle \psi(\sigma), \lambda(\frac{\partial(\psi\sigma)}{\partial u} + k\frac{\partial(\psi\sigma)}{\partial v}) + \frac{\partial(\psi\sigma)}{\partial v}\rangle \subset \pi$ different from $T_{C_{P(\sigma)},\psi(\sigma)}$ meeting $T_{2,\Sigma',P(\sigma)} = \langle \phi(\sigma), \frac{\partial(\phi\sigma)}{\partial v}, \frac{\partial^2(\phi\sigma)}{\partial v^2}\rangle$. Observe that $T_{2,\Sigma',P(\sigma)}$ meets π in a point $R(P(\sigma))$. So there will not be other developable direction (and (Σ, p, q) will be a γ_2–congruence) if and only if $R(P(\sigma))$ belongs to every tangent line to $C_{P(\sigma)}$, and thus these curves are lines passing through $R(P(\sigma)) = \pi \cap T_{2,\Sigma',P(\sigma)}$. ■

β_3/γ_3–**Congruences:** $\dim \Sigma' = 0$**.** In this case all planes of the congruence meet the point Σ', and we get the next characterization:

Theorem 2.2.3 *A β_3/γ_3–congruence consists of the cone, with vertex a point Σ', over a line congruence $\overline{\Sigma}$ in a hyperplane $H \not\ni \Sigma'$. (Σ, p, q) is β_3 iff $\overline{\Sigma}$ is not parabolic; that is, the focal locus on the general line consists of two points. (Σ, p, q) is γ_3 iff $\overline{\Sigma}$ is parabolic; that is, the focal locus on the general line consists of one point.*

Proof: Let $H \subset \mathbf{P}^4$ be a hyperplane such that $\Sigma' \notin H$. We get an isomorphism $\psi : \Sigma \longrightarrow \overline{\Sigma} \subset G(\mathbf{P}^1, H)$, $\sigma \mapsto [\mathbf{P}^2(\sigma) \cap H] = [\mathbf{P}^1(\sigma)]$. To conclude, it's enough to observe that $Q \in \mathbf{P}^1(\sigma)$ is a focal point for the line congruence iff $\langle Q, P(\sigma)\rangle$ is a focal line for Σ. ■

2.3 α–Congruences

Let $U \subset \Sigma$ be the open set such that the focal locus on $\mathbf{P}^2(\sigma)$, $\sigma \in U$, consists of the conic $r(\sigma) \vee r'(\sigma)$. Suppose $r(\sigma)$ is the focal line corresponding to the only

developable system γ passing through $\mathbf{P}^2(\sigma)$. We stratify $U = \bigcup_{i\in I} \gamma_i$, where γ_i are the developable systems contained in the congruence. Let $R(U) := \{(P,\sigma) : P \in r(\sigma)\} = \bigcup F_1(\gamma_i) \subset F_1(U)$ and consider the restrictions $\bar{p} : R(U) \longrightarrow \mathbf{P}^4$, $\bar{q} : R(U) \longrightarrow U$ of the maps p, q.

1. If $\dim \bar{p}(R(U)) = 2$, we get that $p(F_1(\gamma_i))$ must be 1–dimensional by arguing as in theorem 2.1.1. So $\bar{p}(R(U))$ is a ruled surface with generators $p(F_1(\gamma_i))$.

2. If $\dim \bar{p}(R(U)) = 3$, $p(F_1(\gamma_i))$ must be 2–dimensional for the general curve γ_i. So $p(F_1(\gamma_i))$ is a developable ruled surface and γ_i is the corresponding family of tangent planes. In particular, $p(F_1(\gamma_i))$ is a cone if $p(F_2(\gamma_i))$ is a point. Now let $(P,\sigma) \in r(\sigma) \subset F_1(\gamma_i) \subset R(U)$ be a general point. Since $r(\sigma)$ is the focal line in $\mathbf{P}^2(\sigma)$ for the direction $T_{\gamma_i,\sigma}$, we get $\ker(dp)_{(P,\sigma)}|_{T_{R(U),(P,\sigma)}} \subset T\gamma_i, \sigma$. Then, because $(d\bar{p})_{(P,\sigma)} = (dp)_{(P,\sigma)}|_{T_{R(U),(P,\sigma)}}$, (P,σ) is focal in $R(U))$ iff (P,σ) is focal in γ_i, and $F_1(R(U)) = \bigcup F_2(\gamma_i)$; that is, the focal locus of $R(U)$ consists of a point over each general line. Since (Σ, p, q) is not a β_3 or γ_3–congruence, necessarily $\dim p(F_1(R(U))) > 0$. We have two possibilities:
   either $\dim p(F_1(R(U))) = 2$ and $\dim p(F_1(\gamma_i)) = 1$ (γ_i consists of the osculating planes to $p(F_2(\gamma_i))$), or $\dim p(F_1(R(U))) = 1$ and $\dim p(F_1(\gamma_i)) = 0$ (γ_i consists of the tangent planes to the cone $p(F_1(\gamma_i))$).

We have the following cases

α_1–**Congruences:** $\dim p(R(U)) = 3$ and $\dim p(F_1(R(U))) = 2$.

Theorem 2.3.1 (Σ, p, q) *is an α_1–congruence iff it consists of the family of osculating planes to a 1–dimensional family of curves C_i on a nondegenerate and irreducible surface $S \subset \mathbf{P}^4$, except if C_i is asymptotic (Σ is a γ_1–congruence) or $\{C_i\}$ is one of the families of a conjugate double system (Σ is a β_1–congruence).*

Proof: Given an α_2–congruence, the result follows by taking $C_i = p(F_2(\gamma_i))$ on the nondegenerate and irreducible surface $S = p(F_1(R(U)))$. Conversely, given S and C_i on S, let Σ be the congruence defined by the osculating planes. If Σ is not an α–congruence, there is a point in each plane which is a focal point for every direction. Suppose a general point $x \in S$ and let $x(u,v)$ be a parametrization of S around x such that the curves C_i are the solutions of $dv = 0$. We get a parametrization $\sigma(u,v)$ of Σ such that $\mathbf{P}^2(\sigma(u,v)) = \langle x(u,v), x_u(u,v), x_v(u,v)\rangle$. $\langle x, x_u\rangle$ is focal for the direction $\frac{\partial}{\partial u}$. Assume the existence of a point in $\langle x, x_u\rangle$ that is a focal point for $\frac{\partial}{\partial v}$. This is equivalent to the condition $\mathrm{rank}(x, x_u, x_{uu}, x_v, x_{uv}) \leq 4$; that is, there is $(c:d)$ such that $\mathrm{rank}(x, x_u, x_v, cx_{uu} + dx_{uv}) \leq 3$ for every (u,v). This provides, by Remark 2.2.1 the exceptions in the theorem. ■

α_2–**Congruences:** $\dim p(R(U)) = 3$ and $\dim p(F_1(R(U))) = 1$.

Theorem 2.3.2 (Σ, p, q) *is an* α_2*–congruence iff it consists of the tangent planes to a* 1*–dimensional family of cones whose vertexes vary along a curve* $C \subset \mathbf{P}^4$*, and such that the focal locus of the generators of each cone consists of a point on every line.*

Proof: If (Σ, p, q) is an α_2–congruence, we conclude by taking the cones $Q_i := p(F_1(\gamma_i))$ whose vertexes vary along the curve $C := p(F_1(R(U)))$ and the focal locus of the generators of Q_i is a point on each generator. Conversely, given a curve C and a 1–dimensional family of cones Q_i, whose vertexes vary along C, we consider the family Σ of their tangent planes. Since the tangent planes to each cone is a developable system, the congruence is not in cases δ, γ_1 and γ_2; and since the vertexes of the cones vary along a curve, Σ cannot be a β_3 or γ_3 congruence. Then the possibilities are: α, β_1 or β_2.

- If Σ is a β_1–congruence, it consists of the tangent planes to a surface Σ' with a conjugate double system. Consider a parametrization such that the conjugate double system is defined by the equation $(dv)(du) = 0$. The family of lines $R(U)$ is parametrized by $\langle x(u,v), x_u(u,v)\rangle$ and there are two different focal points on each line: the point x for the direction $\frac{\partial}{\partial u}$; and the vertex of the corresponding cone $ax + bx_u$, $b \neq 0$, for the direction $\frac{\partial}{\partial v}$.
- If Σ is a β_2–congruence, there is another different developable system passing through each plane $\mathbf{P}^2(\sigma)$: it consists of planes containing the line $T_{\Sigma', P(\sigma)}$ meeting to $r(\sigma)$ in $P(\sigma)$. We see that Σ' is a directrix for each cone and all of its points are fundamental points, then they are focal points for the family $\{r(\sigma)\}$. Furthermore the vertex of the cone is a focal point on every line, and there are two focal points on every line.

We conclude that Σ is an α_2–congruence. ■

α_3–**Congruences:** $\dim p(R(U)) = 2$.

Theorem 2.3.3 (Σ, p, q) *is an* α_3*–congruence iff it consists of a* 1*–dimensional family of planes passing through every generator of a nondegenerate and irreducible ruled surface* S *except the tangent planes to* S*, which is a* γ_1*–congruence.*

Proof: If (Σ, p, q) is a α_3–congruence, $S := \overline{p(R(U))}$ is a ruled surface with generators $r_i := p(F_1(\gamma_i))$ and there is a 1–dimensional family of planes passing through every generator. S cannot be a developable surface because Σ would be in cases β_2, γ_2, β_3 or γ_3. Conversely, given such a surface $S \subset \mathbf{P}^4$ and the congruence Σ constructed in that way, is not in case δ. Suppose Σ is not an α–congruence and consider $\Sigma' := \overline{p(S(U))} \subset \overline{p(R(U))} = S$. Since S is neither the tangent developable to a curve nor a cone, Σ cannot be in cases β_2/γ_2 and β_3/γ_3. So $\Sigma' = S$ and Σ is the family of tangent planes to S. ■

3 Conclusions.

Criterium 1.3		**Characterization**
α: Only one developable system passing through the general plane.	α_1	Family of osculating planes to a 1–dimensional family of curves on an irreducible nondegenerate surface S ($= p(F_1(R(U)))$), except the case where such curves are asymptotic or they are curves of a conjugate double system of S.
	α_2	Family of tangent planes to a 1-dimensional family of cones with vertexes on a curve ($p(F_1(R(U)))$) such that the set of generators of the cones ($p(R(U))$) has the corresponding vertex as generic focal locus in each line.
	α_3	1–dimensional families of planes containing each of the generators of an irreducible ruled surface ($p(F_1(R(U)))$), nondevelopable, except the family of tangent planes to such a surface.
β: Two different developable systems passing through the general plane.	β_1	Family of tangent planes to an irreducible nondegenerate surface without asymptotic curves ($\Sigma' \subset \mathbf{P}^4$).
	β_2	A 1–dimensional family of planes passing through each one of the tangent lines of a curve (Σ'). If these families are linear they cannot contain the corresponding osculating plane to Σ'.
	β_3	Cone (with vertex a point Σ') of a nonparabolic congruence of lines in a hyperplane $H \subset \mathbf{P}^4$, $\Sigma' \notin H$.
γ: Two coincident developable systems passing through the general plane.	γ_1	Family of tangent planes to an irreducible nondegenerate surface with asymptotic curves ($\Sigma' \subset \mathbf{P}^4$).
	γ_2	A 1–dimensional linear family of planes passing through each one of the tangent lines to a curve (Σ') satisfying that each one of such families must contain the corresponding osculating plane to Σ'.
	γ_3	Cone (with vertex a point Σ') of a parabolic congruence of lines in a hyperplane $H \subset \mathbf{P}^4$, $\Sigma' \notin H$.
δ: An infinity of developable systems passing through the general plane.	δ	Family of the planes in $\mathbf{P}^4$ containing a line ($p(F_1(\Sigma))$).

References

[1] CHIANTINI, L- CILIBERTO, C. A few remarks on the lifting problem. *Asterisque 218. Journées de Geométrie Algébrique D'Orsay.* (1993), 95-109.

[2] CILIBERTO, C. - SERNESI, E. Singularities of the Theta Divisor and Congruences of planes.

Journal of Algebraic Geometry **1** (1992), 231-250.

[3] ITALIANI, M. Sulle congruenze di piani nello spazio proiettivo $\mathbf{S}_4$.

Boll. Un. Mat. it. (3) 13. (1958).

[4] ITALIANI, M. Sulle congruenze di piani nello spazio proiettivo $\mathbf{S}_4$.

Università di Bologna. 1959.

[5] LANE, E.P. *A treatise on projective differential geometry.*

The University of Chicago Press. 1941.

[6] ROGORA, E. Classification of Bertini's series of varieties of dimension less than or equal to four.

Geom. Dedicata **64** (1997) no.2, 157–191.

[7] SEGRE, C. Preliminari di una teoria delle varietà loughi di spazi.

Opere, vol II, p. 71-118. Unione Matematica Italiana. Roma. Ed. Cremonese. 1958.

[8] SEGRE, C. Sui fuochi di secondo ordine dei sistemi infiniti di piani e sulle curve iperspaziali con una doppia infinità di piani plurisecanti.

Opere, vol II, p. 149-153. Unione Matematica Italiana. Roma. Ed. Cremonese. 1958.